CHARACTERIZING FINITE p-GROUPS BY THEIR SCHUR MULTIPLIERS, $t(G) = 5$

PEYMAN NIROOMAND

Abstract. Let G be a finite p-group of order p^n. It is known that $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ and $t(G) \geq 0$. The structure of G characterized when $t(G) \leq 4$ in [1, 5, 13, 15, 18]. The structure description of G is determined in this paper for $t(G) = 5$.

1. introduction

Let G be a finite p-group and $\mathcal{M}(G)$ denotes the Schur multiplier of G. It is known that $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$, where $t(G) \geq 0$ by the result of Green in [8].

The Structure of G is determined when $t(G) = 0, 1$ in [1]. In the case $t(G) = 2$ and 3, Zhou in [18] and Ellis in [5] determined the structure of G, respectively. Recently all finite p-group G when $t(G) = 4$ are listed in [13] by the author. In the present paper, structure of all finite non-abelian p-groups will be given when $t(G) = 5$. Our method is quite different to that of [1, 5, 18] and depends on the results of [11, 12].

2. Notations and preparatory results

We use notations and terminology of [5, 13]. In this paper, D_8 and Q_8 denote the dihedral and quaternion group of order 8, E_1 and E_2 denote the extra special p-groups of order p^3 of exponent p and p^2, respectively. E_4 denotes the unique central product of a cyclic group of order p^2 and a non-abelian group of order p^3. Also $\mathbb{Z}_{p^n}^{(m)}$ denotes the direct product of m copies of the cyclic group of order p^n. We say that G has the property $t(G) = 5$ or briefly with $t(G) = 5$ if the order its Schur multiplier is equal to $p^{\frac{1}{2}n(n-1)-5}$.

We state some essential theorems which play important roles in the proof of our Main Theorem, without proof as follows.

Theorem 2.1. (See [11, Main Theorem]). *Let G be a non-abelian finite p-group of order p^n. If $|G^{'}| = p^k$, then we have*

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$

In particular,

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1},$$

and the equality holds in this last bound if and only if $G = E_1 \times Z$, where Z is an elementary abelian p-group.

Key words and phrases. Schur multiplier, p-group.

Mathematics Subject Classification 2010. Primary 20D15; Secondary 20E34, 20F18.

Theorem 2.2. (See [10, Theorem 2.2.10]). *For every finite groups H and K, we have*
$$\mathcal{M}(H \times K) \cong \mathcal{M}(H) \times \mathcal{M}(K) \times \frac{H}{H'} \otimes \frac{K}{K'}.$$

Theorem 2.3. (See [10, Theorem 3.3.6]). *Let G be an extra special p-group of order p^{2m+1}. Then*

(i) *If $m \geq 2$, then $|\mathcal{M}(G)| = p^{2m^2-m-1}$.*
(ii) *If $m = 1$, then the order of Schur multipliers of D_8, Q_8, E_1 and E_2 are equal to $2, 1, p^2$ and 1, respectively.*

3. Main Theorem

In this section we intend to characterize all finite non-abelian p-groups with the property $t(G) = 5$. In fact, we have

Theorem 3.1 (Main Theorem)**.** *Let G be a non-abelian p-group of order p^n. Then*
$$|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-5}$$
if and only if G is isomorphic to one of the following groups.

(1) $D_8 \times \mathbb{Z}_2^{(3)}$,
(2) $E_1 \times \mathbb{Z}_p^{(4)}$,
(3) $E_2 \times \mathbb{Z}_p^{(2)}$,
(4) $E_4 \times \mathbb{Z}_p$,
(5) *extra special p-group of order p^5,*
(6) $\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p\rangle$,
(7) $\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b] = 1\rangle$,
(8) $\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = 1, [a, b, b] = a^{np}, [a, b, b, b] = 1\rangle$, *where n is a fixed quadratic non-residue of p and $p \neq 3$,*
(9) $\langle a, b \mid a^{p^2} = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1\rangle$,
(10) $\langle a, b \mid a^p = 1, b^p = [a, b, b], [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1\rangle$,
(11) D_{16},
(12) $\langle a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^{-1}\rangle$,
(13) $Q_8 \times \mathbb{Z}_2^{(2)}$,
(14) $(D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
(15) $(Q_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$,
(16) $\mathbb{Z}_2 \times \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$.

We separate the proof of it into several steps as follows.

Lemma 3.2. *Let G be a p-group of order p^n and $|G'| = p^k (k \geq 2)$ with $t(G) = 5$. Then $n \leq 4$ unless $k = 2$, in this case $n \leq 6$.*

Proof. By virtue of Theorem 2.1, we have
$$\frac{1}{2}(n^2 - n - 10) \leq \frac{1}{2}(n + k - 2)(n - k - 1) + 1 \leq \frac{1}{2}n(n - 3) + 1,$$
which follows the result. □

Theorem 3.3. *Let G be a non-abelian finite p-group of order p^n with $t(G) = 5$. Then $|G| \leq p^7$. In the case that $n = 6$ and $n = 7$, G is isomorphic to*
$$D_8 \times \mathbb{Z}_2^{(3)} \text{ and } E_1 \times \mathbb{Z}_p^{(4)},$$

respectively.

Proof. One can easily check that $n \leq 7$ by Theorem 2.1.

In the case $n = 7$, Lemma 3.2 follows that $|G^{'}| = p$. Since $|\mathcal{M}(G)| = p^{16}$ and equality holds in Theorem 2.1, we should have $G \cong E_1 \times \mathbb{Z}_p^{(4)}$. When $n = 6$, $|\mathcal{M}(G)| = p^{10}$ and by a consequence of [12, Main Theorem], we have $G \cong D_8 \times \mathbb{Z}_2^{(3)}$. □

As mentioned in the Lemma 3.2 and Theorem 3.3, we may assume that $n \leq 5$. First assume that $p \neq 2$.

Theorem 3.4. *Let $|G| = p^5$ $(p \neq 2)$ and $|G^{'}| \geq p^2$. Then there is no group with $t(G) = 5$.*

Proof. Using Lemma 3.2, we may assume that $|G^{'}| = p^2$.

For each central subgroup K of order p, [10, Theorem 4.1] implies that

$$p^5 = |\mathcal{M}(G)| \leq p^2 \ |\mathcal{M}(G/K)|.$$

If for every central subgroup K, $|\mathcal{M}(G/K)| = p^4$ the proof of [12, Main Theorem] shows that $G \cong \mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p$ and hence $|\mathcal{M}(G)| = p^6$, which is a contradiction. Thus there exists a central subgroup K such that $|\mathcal{M}(G/K)| \leq p^3$. Since $p \neq 2$ and $|G/K| = p^4$, [12, Main Theorem] follows that $|\mathcal{M}(G/K)| \leq p^2$, and so $|\mathcal{M}(G)| \leq p^4$, which contradicts the assumption. □

Theorem 3.5. *Let $|G| = p^5$ $(p \neq 2)$ and $|Z(G)| = p^3$ with $t(G) = 5$, then G is isomorphic to*

$$E_2 \times \mathbb{Z}_p^{(2)} \text{ or } E_4 \times \mathbb{Z}_p.$$

Proof. It is known by [10, Theorem 4.1] that,

$$|\mathcal{M}(G)||G^{'}| \leq |\mathcal{M}(G/G^{'})||G^{'} \otimes G/Z(G)|.$$

We know that $|G^{'}| = p$ by Theorem 3.4. Now, if $G/G^{'}$ is not elementary abelian, then $|\mathcal{M}(G/G^{'})| \leq p^3$, and so $|\mathcal{M}(G)| \leq p^4$, which is a Impossible. Therefore, $G/G^{'}$ is elementary abelian. On the other hand, [9, Theorem 2.2] implies that $Z(G)$ is of exponent at most p^2. Thus two cases may be considered.

Case I. First suppose that $Z(G)$ is of exponent p. By a result of [11, Lemma 2.1], we should have $G \cong H \times \mathbb{Z}_p^{(2)}$, where H is extra special of order p^3. Since $|\mathcal{M}(G)| = p^5$, Theorems 2.2 and 2.3 imply that $H \cong E_2$.

Case II. In this case, similar to pervious part one can see that $G \cong H \times \mathbb{Z}_{p^2}$, where H is extra special of order p^3 or $G \cong E_4 \times \mathbb{Z}_p$. By invoking Theorems 2.2 and 2.3, the order of the Schur multiplier of $H \times \mathbb{Z}_{p^2}$ is at most p^4, and hence does not have the property $t(G) = 5$. On the other hand, by a result of [13, Lemma 3.5] and Theorem 2.2, we should have $|\mathcal{M}(E_4 \times \mathbb{Z}_p)| = p^5$, as required. □

Theorem 3.6. *Let $|G| = p^5$ $(p \neq 2)$ and $|Z(G)| = p^2$. Then there is no group with $t(G) = 5$.*

Proof. We may assume that $G/G^{'}$ is not elementary abelian by appealing to [11, Lemma 2.1]. Using [6, Proposition 1], $G/Z(G)$ is elementary abelian and $G/G^{'} \cong \mathbb{Z}_2^{(2)} \times \mathbb{Z}_2$. Hence $Z(G)$ and Frattini subgroup coincide, and so [6, Proposition 1] (see also [4, Proposition 5 (i) and (ii)]) shows that

$$p^2|\mathcal{M}(G)| \leq |\mathcal{M}(G/G^{'})||G^{'} \otimes G/Z(G)| \leq p^6.$$

Thus $|\mathcal{M}(G)| \leq p^4$, which is a contradiction. $\square$

Lemma 3.7. *Every extra special p-group of order p^5 has the property $t(G) = 5$.*

Proof. It is straightforward by Theorem 2.3. $\square$

Theorem 3.8. *Let $|G| = p^4$ and $|G^{'}| = p$ with $t(G) = 5$. Then G is isomorphic to*

$$\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle.$$

Proof. First suppose that $G/G^{'}$ is elementary. By a result of [11, Lemma 2.1], we have $G \cong H \times \mathbb{Z}_p$ or $G \cong E_4$. The order of Schur multipliers of both of them is at least p^2. Thus $G/G^{'}$ can not be elementary abelian. Since G^p and $G^{'}$ are contained in $Z(G)$, we consider two cases.

Case I. First assume that $G^{'} \cap G^p = 1$, then $G/G^p \cong E_1$, and so $|\mathcal{M}(G)| \geq |\mathcal{M}(E_1)| = p^2$ directly by using [10, Corollary 2.5.3 (i)].

Case II. In this case, we have two possibilities for $Z(G)$. The first possibility is $Z(G) = G^p \cong \mathbb{Z}_{p^2}$, thus G is of exponent p^3 and obviously $|\mathcal{M}(G)| = 1$. The second possibility is $Z(G) = G^p \cong \mathbb{Z}_p \times G^{'}$. By [2, pp. 87-88], there is a unique group of order p^4 with this properties, which is isomorphic to

$$\langle a, b \mid a^{p^2} = 1, b^{p^2} = 1, [a, b, a] = [a, b, b] = 1, [a, b] = a^p \rangle.$$

$\square$

Lemma 3.9. *Let $|G| = p^4$ and $|G^{'}| = p^2$ with $t(G) = 5$. Then G is isomorphic to*

$$\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = [a, b, b] = a^p, [a, b, b, b] = 1 \rangle,$$

$$\langle a, b \mid a^{p^2} = b^p = 1, [a, b, a] = 1, [a, b, b] = a^{np}, [a, b, b, b] = 1 \rangle,$$

where n is a fixed quadratic non-residue of p and $p \neq 3$,

$$\langle a, b \mid a^{p^2} = 1, b^3 = a^3, [a, b, a] = 1, [a, b, b] = a^6, [a, b, b, b] = 1 \rangle,$$

$$\langle a, b \mid a^p = 1, b^p = [a, b, b], [a, b, a] = [a, b, b, a] = [a, b, b, b] = 1 \rangle.$$

Proof. The result is obtained from [5, pp. 4177] and [2, pp. 88] (see also [16, pp. 196-198]). $\square$

Lemma 3.10. *Let G be a p-group of order 16 with $t(G) = 5$. Then G is isomorphic to*

$$D_{16} \text{ or } \langle a, b \mid a^4 = b^4 = 1, a^{-1}ba = b^{-1} \rangle.$$

Proof. See table I on [14]. $\square$

Lemma 3.11. *Let G be a p-group of order 32 with $t(G) = 5$. Then G is isomorphic to*

$$Q_8 \times \mathbb{Z}_2^{(2)}, (D_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2, (Q_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \text{ or}$$

$$\mathbb{Z}_2 \times \langle a, b, c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle.$$

Proof. These groups are obtained by using the HAP package [7] of GAP [17]. $\square$

References

[1] Berkovich, Ya. G. (1991). On the order of the commutator subgroups and the Schur multiplier of a finite p-group. *J. Algebra* 144: 269272.
[2] Burnside, W. (1955). *Theory of Groups of Finite Order*. New York: Dover Publications.
[3] Brown, R., Johnson, D. L., Robertson, E. F. (1987). Some computations of non-abelian tensor products of groups. *J. Algebra* 111: 177–202.
[4] Ellis, G. (1998). A bound for the derived and Frattini subgroups of a prime-power group. *Proc. Amer. Math. Soc.* (126)9: 2513-2523.
[5] Ellis, G. (1999). On the Schur multiplier of p-groups. *Comm. Algebra* 27(9): 4173-4177.
[6] Ellis, G., Wiegold, J. (1999). A bound on the Schur multiplier of a prime-power group. *Bull. Austral. Math. Soc.* 60: 191-196.
[7] Ellis, G. (2008). *HAP-Homological algebra programming.* A refreed GAP 4 package (GAP Group 2005), available at http://hamilton.nuigalway.ie/Hap/www.
[8] Green, J. A. (1956). On the number of automorphisms of a finite group.*Proc. Roy. Soc.* A 237: 574-581.
[9] Jones, M. R. (1972). Multiplicators of p-groups. *Math. Z.* 127: 165-166. (1973) 450456.
[10] G. Karpilovsky, The Schur multiplier, London Math. Soc. Monogr. (N.S.) 2 (1987).
[11] Niroomand, P. (2009). On the order of Schur multiplier of non-abelian p-groups. *J. Algebra* 322: 4479–4482.
[12] Niroomand, P. A note on the Schur multiplier of prime power groups, submitted.
[13] Niroomand, P. Characterizing finite p-groups by their Schur multipliers, submitted.
[14] Niroomand, P., Rezaei, R. On the exterior degree of finite groups. *Comm. Algebra*, to appear.
[15] Salemkar, A. R., Moghaddam, M. R. R., Davarpanah, M., Saeedi, F.(2007). A remark on the Schur multiplier of p-groups. *Comm. Algebra* 35: 1215-1221.
[16] Schenkman, E. (1975). *Group theory.* Corrected reprint of the 1965 edition. Robert E. Krieger Publishing Co., Huntington, N.Y.
[17] The GAP Group. (2005). *GAP–Groups Algorithms and Programming.* version 4.4. Available at: http://www.gap-system.org.
[18] Zhou, X. (1994). On the order of Schur multipliers of finite p-groups. *Comm. Algebra* 22(1): 1-8.

School of Mathematics and Computer Science, Damghan University of Basic Sciences, Damghan, Iran
E-mail address: p_niroomand@yahoo.com